How Many Ways to Slice a Donut?

n-torus separation with graphs

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The Problem

In how many ways can you cut an n-torus such that the cut separates the surface into k pieces, and no excess cuts are made?

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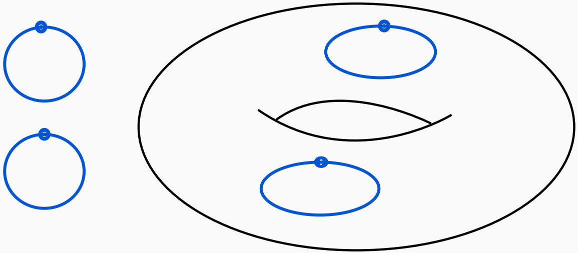

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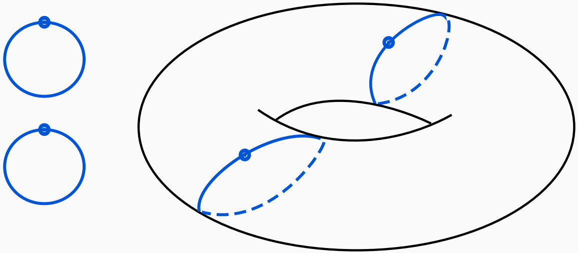

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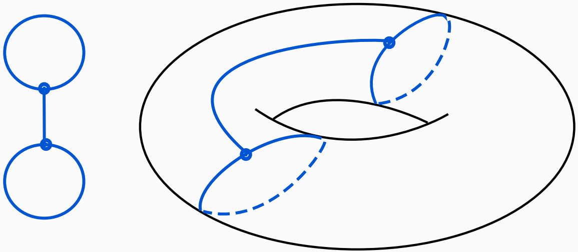

Formalizing the Problem

Definition

A *minimum k-cut of the n-torus* is a multigraph embedding such that

i. It divides the surface into k regions

ii. All vertices and edges in the graph are necessary.

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Problem

How many such minimum k-cuts are there for the n-torus (abbreviated $\min(n, k)$)?

Strategy

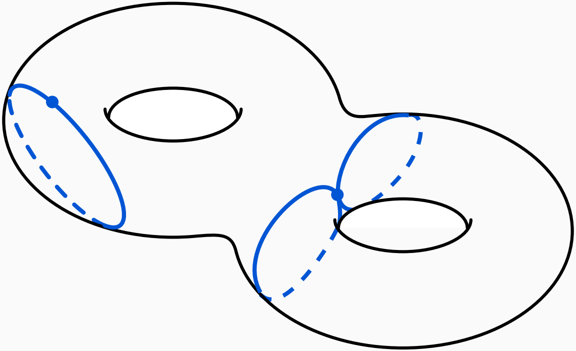
Given the n-torus, we will look over all possible graphs G and, for each graph, determine whether there exists an embedding of that graph that produces a minimum k-cut.

In other words, if $\mathcal{G}$ is the set of all multigraphs,

$$\min(n, k) = \sum_{G \in \mathcal{G}} f(G),$$

where $f(G) = \begin{cases} 0 & G \text{ does not admit a min k-cut} \\ 1 & G \text{ admits a min k-cut} \end{cases}$

Strategy



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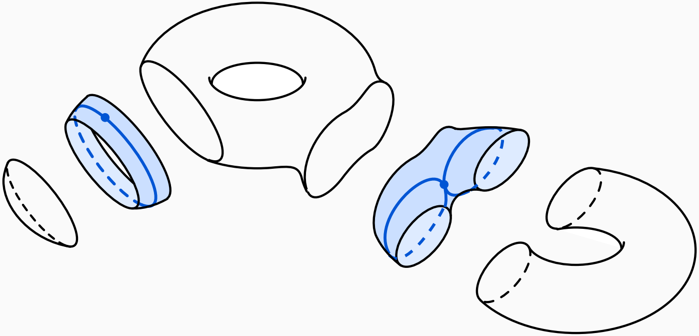

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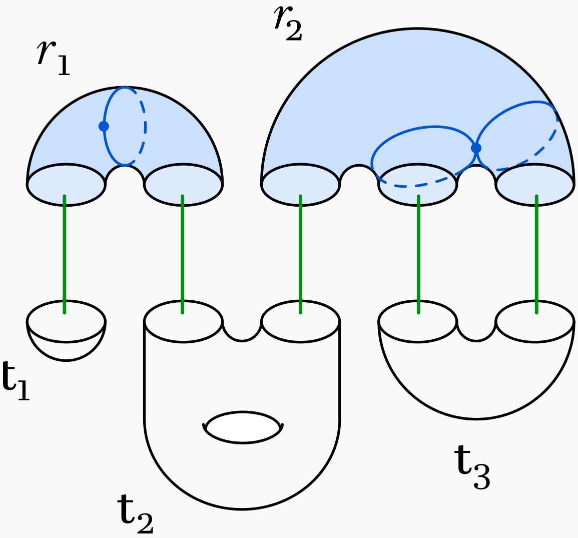
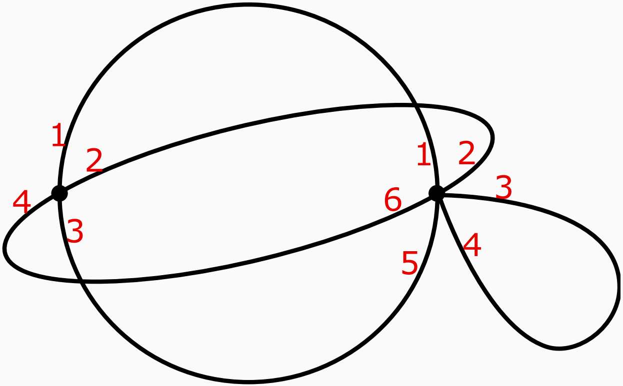


Definition

A *rotation system* is a graph with an ordering of the half-edges about each vertex.



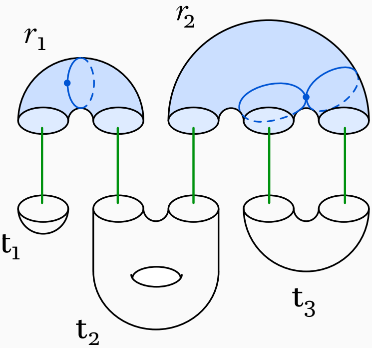
Face Tracing Algorithm

1. Replace edges with directed edges.
2. Determine cycles by following edges to vertices to other edges etc., according to ordering at each vertex.
3. Embed an annulus along each cycle.

Gluing Plan

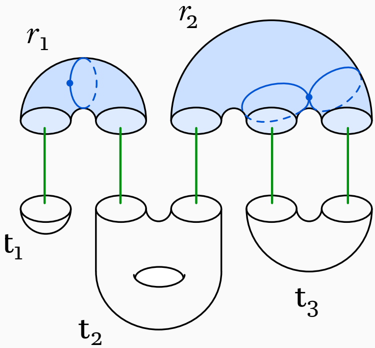
Definition

A *gluing plan* (R, G) is a rotation system R along with a bipartite graph $G = (V, E)$, where V is the set of connected components of R and regions of the torus T, and E is the list of boundary identifications.



Is a Gluing Plan a $\min(n, k)$-cut?

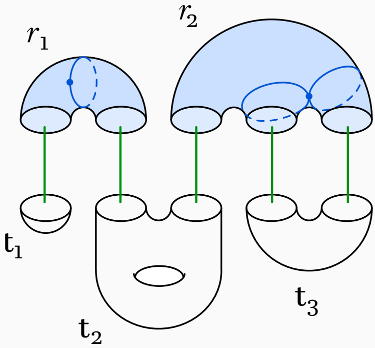
- Are there k torus pieces?
- Does it have n genera?
- Are all vertices and edges necessary?



Counting genera in gluing plans

There are three sources of genera in (R, G):

1. Genera in regions $t_1, \dots, t_m$ of the n-torus T
2. Genera in connected components $r_1, \dots, r_\ell$ of R
3. Genera introduced by gluing

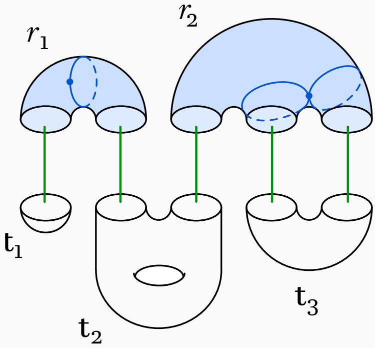


Counting genera in gluing plans

There are three sources of genera in (R, G):

1. Genera in regions $t_1, \ldots, t_m$ of the partitioned n-torus T:

$$g_T = \sum_{t_i \in T} g_{t_i}$$



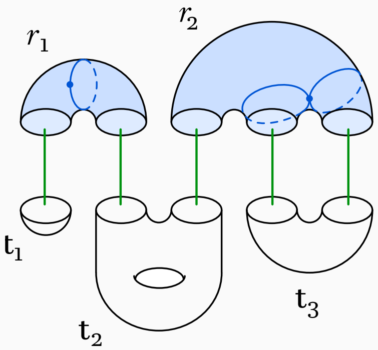
Counting genera in gluing plans

There are three sources of genera in (R, G):

2. Genera in connected components $r_1, \ldots, r_\ell$ of R:

$$g_R = \ell - \frac{1}{2}(e - v - c),$$

where c is the number of cycles in R.



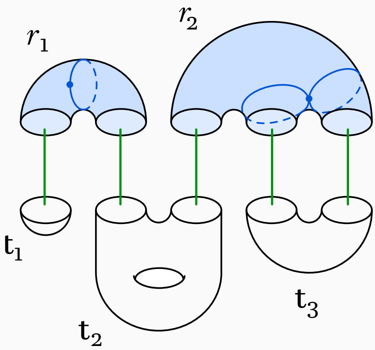
Counting genera in gluing plans

There are three sources of genera in (R, G):

3. Genera introduced by gluing:

$$g_{\text{glue}} = c - (k + \ell) + 1,$$

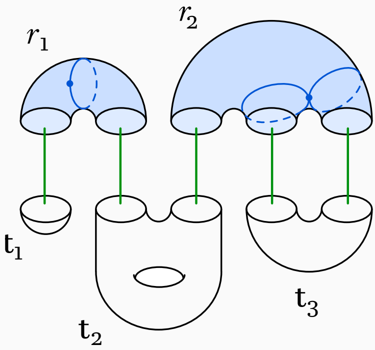
where c is the number of cycles in R.



Counting genera in gluing plans

In summary,

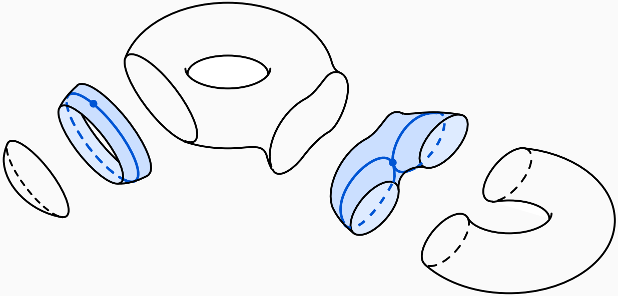
$$g = g_T + g_R + g_{\text{glue}}$$
$$= \frac{1}{2}(c + e - v) - k + 1.$$



Determining minimality

Lemma

An edge is necessary iff it borders different t_is.



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Lemma

A gluing plan is minimal iff the cycle dual of R is k-partite.

Looking forward

- Generative approach to counting rotation systems with k-partite cycle duals.
- Data generation for small n, k.