Knot Embeddings in Improper Foldings

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September 6, 2018

7OSME
Oxford, UK

Overview

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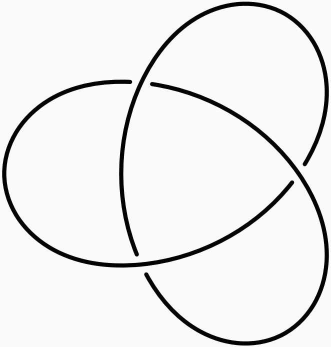
- **Proper foldings**, which are physically realizable and never admit knots, and
- **Improper foldings**, which require self-intersection of the paper, but which can map simple loops in the paper to arbitrary knots in the image.
- Improper foldings allow us to measure the complexity of knots using crease patterns, as captured by the **fold number**.

A quick refresher...

Definition

A **knot** is an embedding of the circle in $\mathbb{R}^3$.



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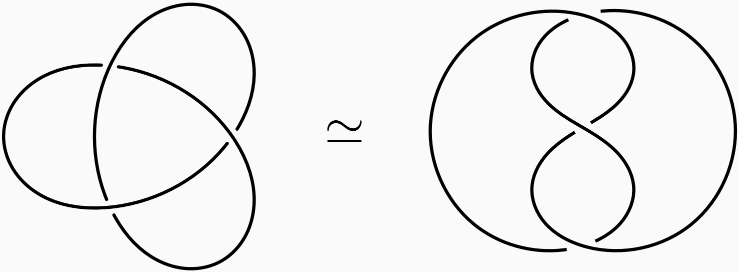

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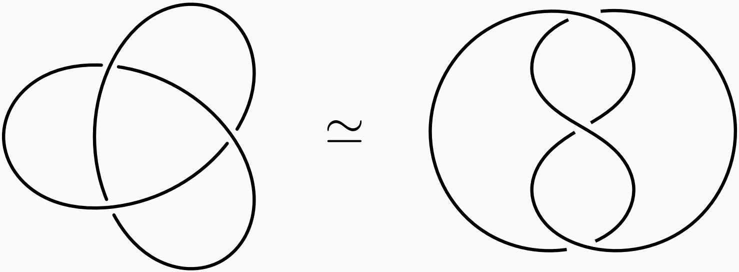

If a knot is equivalent to a planar circle, we call it a trivial knot, or an **unknot**.

Motivation

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Jacques Justin conjectured[1]:

> *"The set of the Jordan curves which are the boundary of F constitutes a link or knot equivalent to a trivial one."*

[1] "Towards a Mathematical Theory of Origami" (1997)

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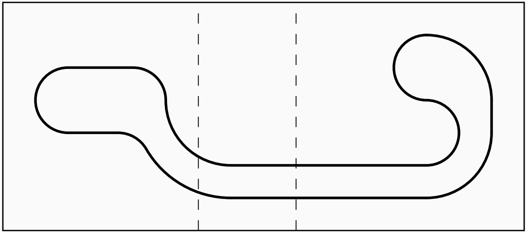

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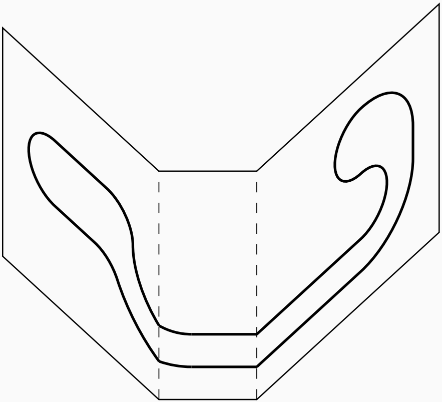

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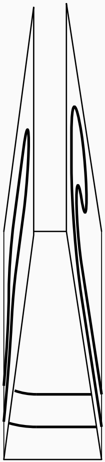

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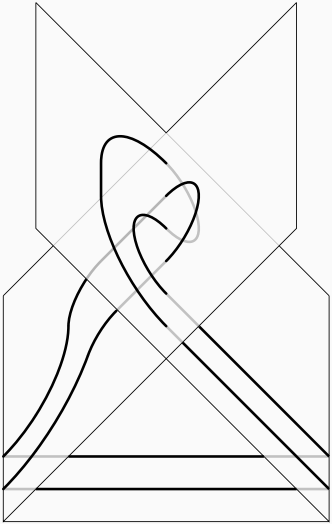

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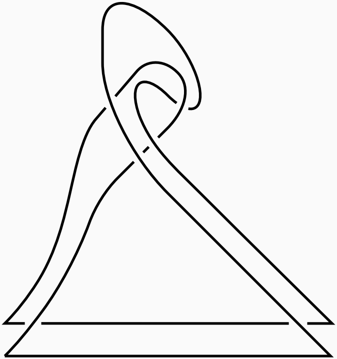

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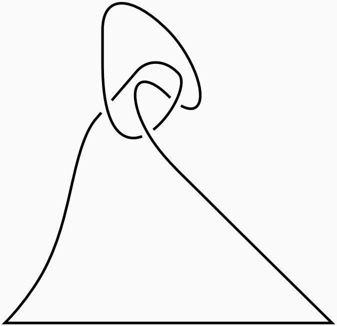

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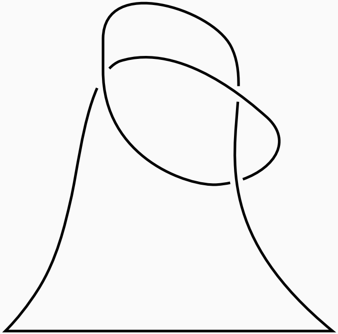

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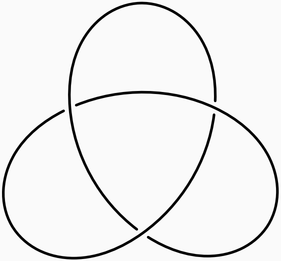

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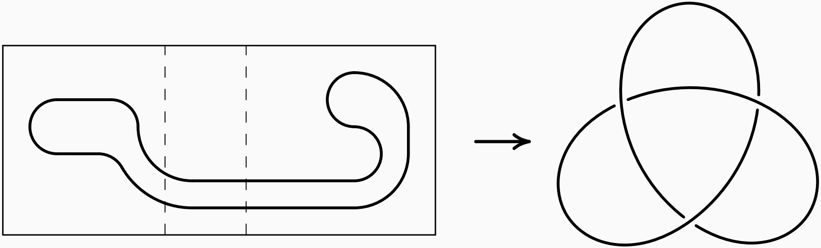

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A **crease pattern** is a graph embedding $G \subset [0, 1]^2$ such that F is non-differentiable precisely on G.

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A knot K is an **origami knot** if there exists a piecewise-linear loop $\ell : S^1 \hookrightarrow [0,1]^2$ on the origami paper and an origami folding F such that $F(\ell) = K$. When this property is satisfied, we say F **admits** K.

Universality & Invariants

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Theorem (Universality)

Every knot type includes an origami knot.

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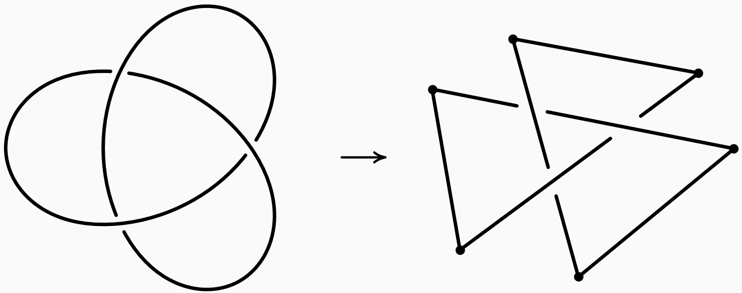

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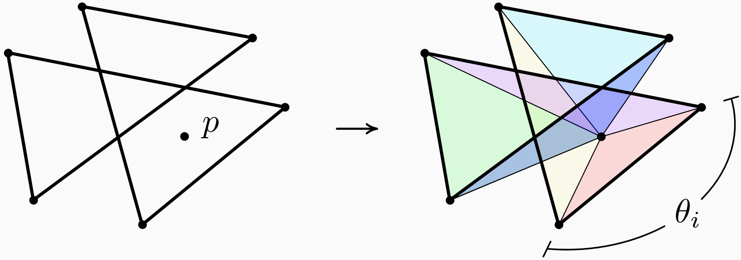

Pick point p in an enclosed region. Note $\sum \theta_i \geq 2\pi$.

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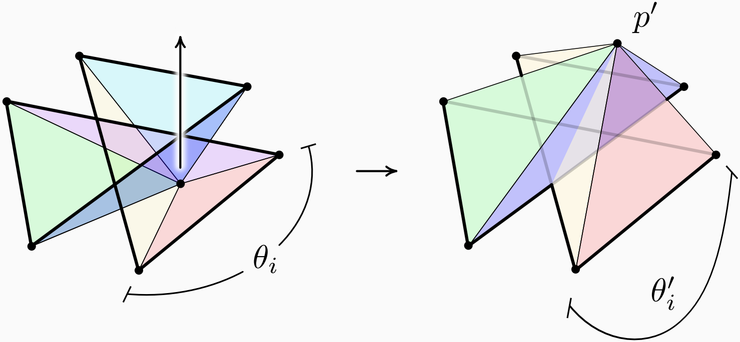

Translate p to p' s.t. $\sum \theta'_i = 2\pi$.

(Intermediate value theorem guarantees the existence of p')

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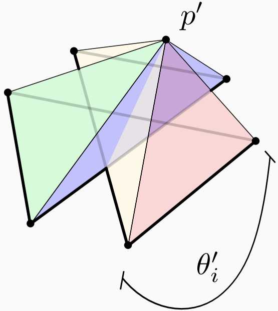

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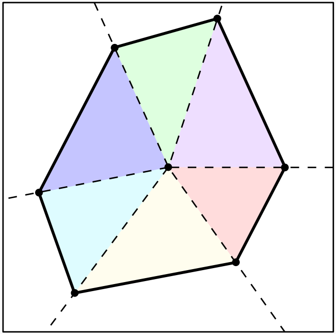

Lay out triangles on the plane.

(Always possible because they sweep out 2π. Choose mountain or valley depending on left- or right-turn.)

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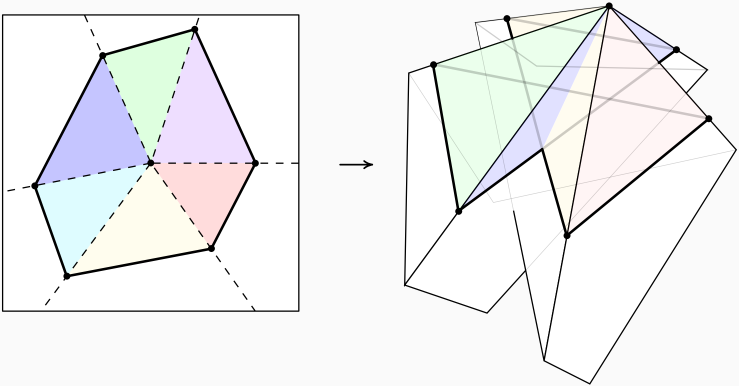

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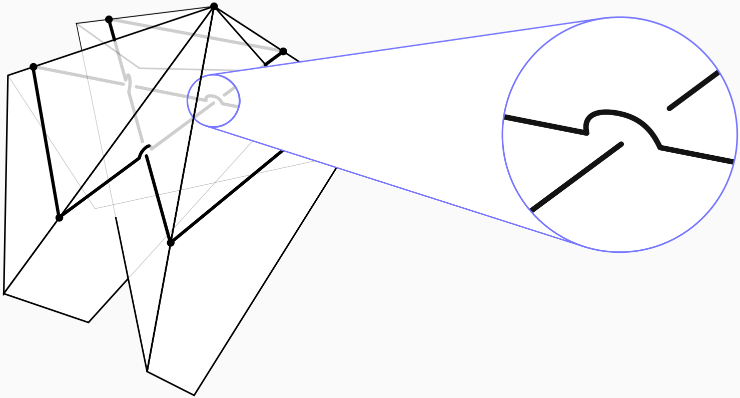

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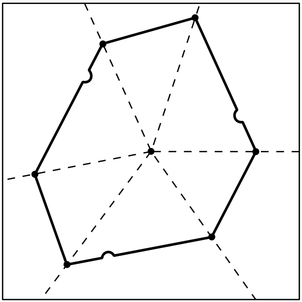

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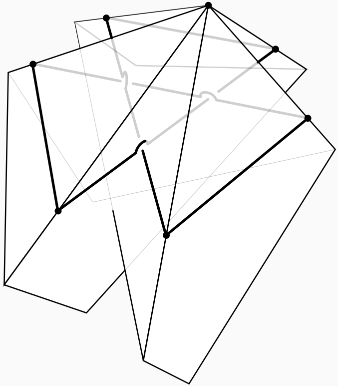

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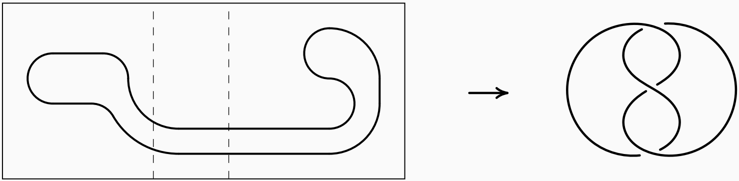

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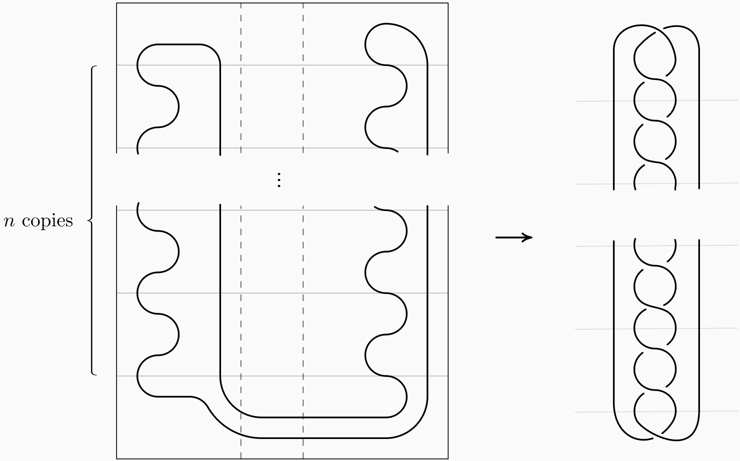
For $n \geq 0$, $(2, 2n+3)$-tori knots have fold number 2.



Proper Foldings

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Intuition:

"If I can fold it with real paper,
it cannot admit a nontrivial knot."

Modeling “real paper”

Pre-existing definition from Justin[2], refined by Demaine & O’Rourke [3], but we’ll pursue a more topological formalization.

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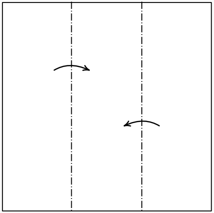

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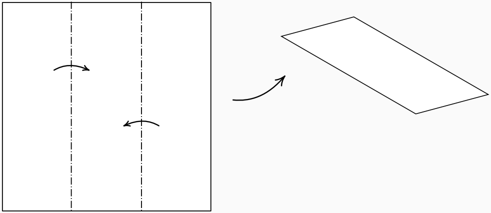

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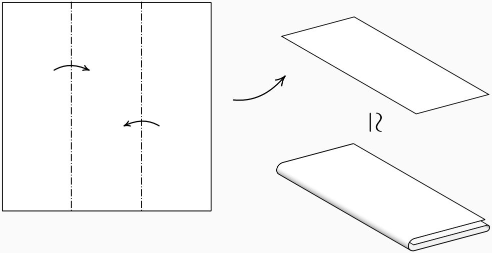

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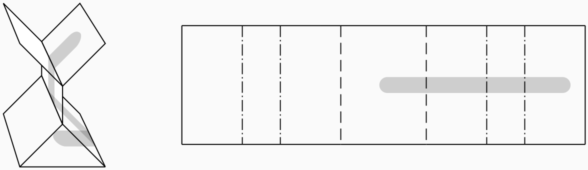

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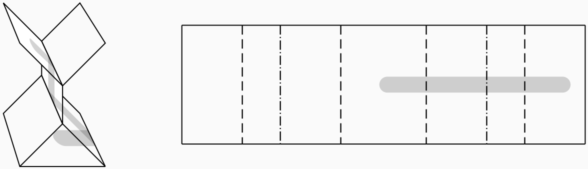

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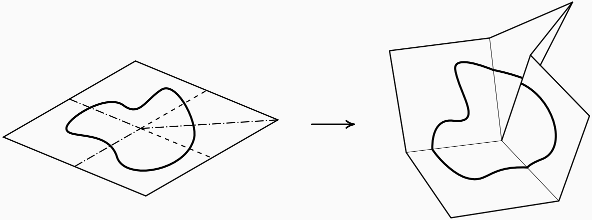

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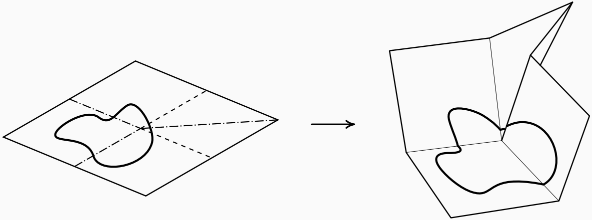

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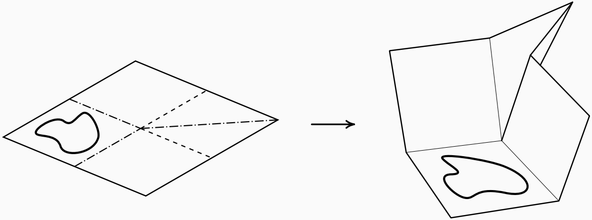

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Consider $G'(\ell)$. If $G'(\ell)$ is not injective, perturb ℓ to an ℓ' such that $G'(\ell')$ is injective. Then $G'(\ell)$ (or $G(\ell')$) is a satellite knot with nontrivial companion K and so $G' \in T^c$. □

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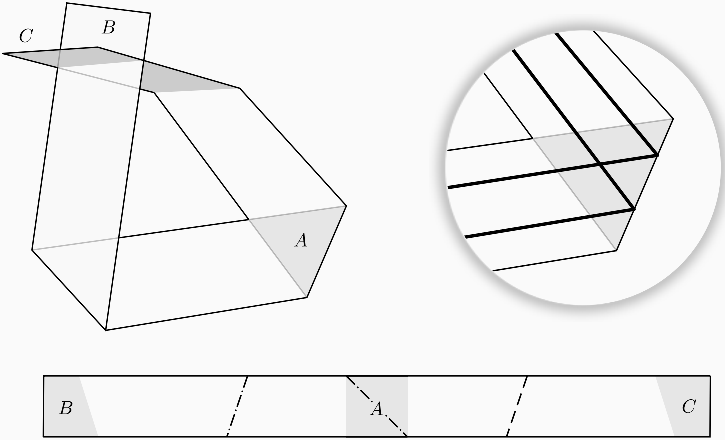
Other Questions:

- **Links:** begin with multiple non-intersecting simple loops in the square and examine foldings which intertwine them.
- **Other topologies:** Punctured disks? Higher dimensions?

Questions

Supplementary Figures



(Counterexample to the converse of Theorem 2)