Operadic Techniques for Meander Enumeration (WIP, 12/28/2019)

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Abstract

Closed meanders are classes of Jordan curves intersecting the x-axis n times, whose enumeration is an important open problem. We present a counting technique based on the algebraic structure arising from a simple meander composition operator. This operad structure allows each meander to be decomposed into a canonical tree form which in turn can be counted with familiar techniques. We demonstrate the utility of this approach by counting a finitely-generated subclass of closed meanders,ultimately showing it to be equinumerous with "blobs of odd order," a previously unrelated combinatorial family. In pursuit of a full enumeration of closed meanders, we show an extension of the composition operator yields a structure on meanders where a meander is indecomposable if and only if it corresponds to a simple permutation. The set of these indecomposable meanders is of independent interest.

Keywords: Meanders, Enumerative Combinatorics, Operads, Simple Permutations

1. Introduction

Closed meanders have a number equivalent definitions (as arch systems, as homeomorphism classes, see [1]); we define them here by their permutations.

Definition 1 (Meander). *Fix a positive integer n and let $m = (1\ x_2\ \dots\ x_{2n})$ denote a cyclic permutation of order $2n$. Define the* upper arches *of m as the*

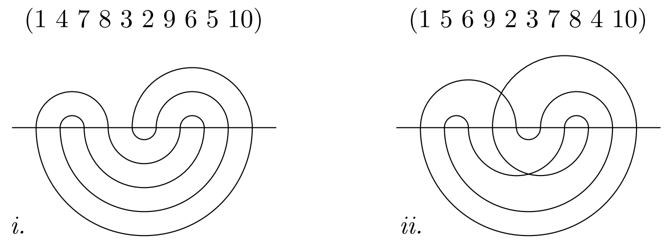

Figure 1: The first cyclic permutation (*i*) is a meander while the second (*ii*) fails to satisfy the ordering conditions on both the upper and lower arches.

set of pairs $U = \{\{x_i, x_{i+1}\} : i \text{ odd}\}$ *and the* lower arches *of* m *as the set of pairs* $L = \{\{x_i, x_{i+1}\} : i \text{ even}\}$.

Then m *is a meander if for every two upper arches (resp. lower arches)* $\{a < a'\}, \{b < b'\}$ *with* $a < b$, *we have* $a' < b$ *or* $a' > b'$.

Graphically, upper (resp. lower) arch pairs $\{a, a'\}$ are represented as semicircles in the plane above (resp. below) the x-axis with endpoints coincident to the points $(a, 0)$ and $(a', 0)$. The ordering conditions correspond to the requirement that no two arches may intersect, as exemplified in Fig. 1.

In the foregoing we will use "meander" to refer to the meander's cyclic permutation $(1\ x_2\ \ldots\ x_{2n})$, the formal sequence of integers $[x_1 = 1, x_2, \ldots, x_{2n}]$, and the corresponding plane curve; we will denote the set of meanders of order n as M_n. Counting M_n is a long-studied problem, appearing to have been initiated in [TODO: date, cite], yet a closed-form enumeration—or even a generating function—has evaded derivation. Many enumeration techniques, both algebraic and asymptotic, have been employed in the literature, including mappings to the Temperley-Lieb algebra [2], Pillowcase covers [TODO: cite], and [TODO: add third technique].

In this article we introduce another approach based on a natural composition operator that allows meanders to be inserted into each other to produce meanders of larger order. In this way, meanders may be rewritten as a composition of smaller meanders, represented by colored trees with certain algebraic properties known as operads. As we shall see, operadic representations of meanders allow

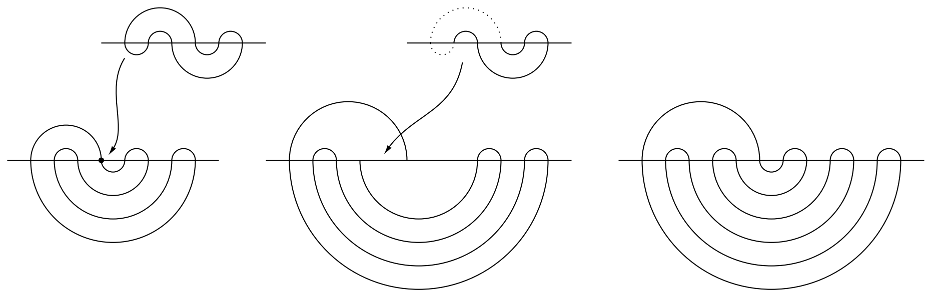

Figure 2: $m \circ_2 w$ for two meanders m, w. If w were inserted at an odd index, it would be reflected about the x axis before insertion.

the development of various recurrence relations for subclasses meanders.

2. The Operad Meander

In this section we define the plain operad MEANDER. MEANDER has no operators of even arity and there is one $(2n-1)$-ary operator for each meander of order n. Composition of operators is defined directly in terms of meander permutations.

Definition 2 (Meander Composition)**.** *Suppose* $m = [1, m_2, \ldots, m_{2n}]$, $w = [1, w_2, \ldots, w_{2k}]$ *are meanders and* i *is an integer with* $1 < i \leq 2n$. *Define the sequences* $m' = m'_1, \ldots, m'_{2n}$ *and* $w' = w'_1, \ldots, w'_{2k}$ *with entries*

$$m'_j = \begin{cases} m_j & \text{if } m_j \leq m_i \\ m_j + 2(k-1) & \text{if } m_j > m_i \end{cases} \qquad \text{and} \qquad w'_\ell = w_\ell + m_i - 2$$

for $1 \leq j \leq 2n$ *and* $1 \leq \ell \leq 2k$. *Then the composition of* w *into* m *at* i *is the meander with permutation*

$$m \circ_i w = [m'_1, \ldots, m'_{i-1}, w'_2, \ldots, w'_{2k}, m'_{i+1}, \ldots, m'_{2n}].$$

Graphically, this corresponds to deleting the two leftmost arches in w, splitting the arches apart in m that meet at the i^{th} intersection (as indexed along the meander curve), and fitting w into the newly created space in m while preserving orientation (see Fig. 2).

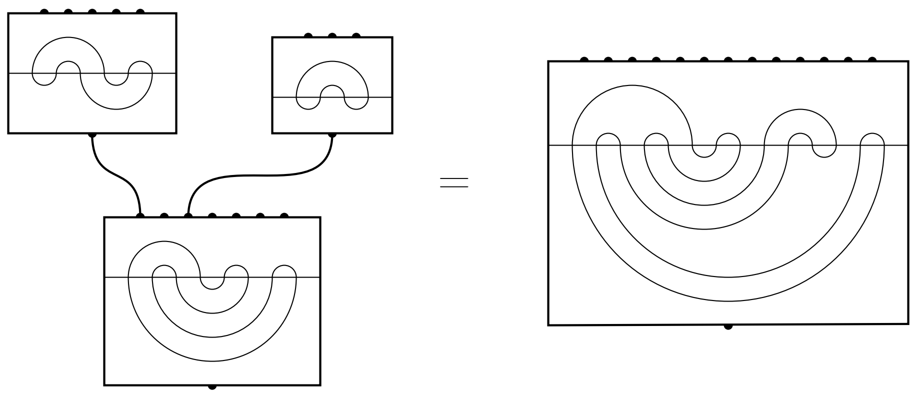

Figure 3: A graphical representation of the insertion operator.

We check a few properties.

Proposition 1. *Composition is well-defined. That is, if m and w are meanders, $m \circ_i w$ is a meander of the correct order.*

Proof. The adjustments m' an w' do not change the relative ordering of the indices, so we need only consider potential violations between an arch from m and an arch from w. But by construction all the inserted arches from w are located either completely inside, or completely outside, each arch from m.

Suppose m and w have orders n and k respectively. Then m is a $(2n-1)$-ary operator and w is a $(2k-1)$-ary operator. By the axiom of operad composition $m \circ_i w$ has arity $2n+2k-3$. We must verify then that $m \circ_i w$ has order $n+k-1$. This is easy to check. □

Proposition 2. *Composition order does not matter. That is, for $i \neq j$, $(m \circ_i w) \circ_j v = (m \circ_j v) \circ_i w$.*

Proof. Exercise left to reader. □

Therefore $\circ_i$ can be extended to $\circ = (\circ_1, \ldots, \circ_n)$. It's easy to check that associativity holds and that the meander of order one, (1 2), is the identity for $\circ$. This completes the definition of the operad MEANDER. See Fig. 3 for an example composition.

3. Basic facts about Meander

The power of an operadic approach to enumeration derives from the ability to write meanders as compositions of smaller meanders. These decompositions are not only equivalent but *equal* to the resulting meander, and as such can be viewed as notational artifacts of the algebraic properties of operads. However, we will treat these *meander expressions* as first-class citizens, as their treelike form admits easy recurrence relations. We give a few facts about these meander representations now.

As shown in Fig. [make fig], some meanders have multiple representations in MEANDER, while other meanders have one representation. It's easy to characterize which meanders have this property:

Proposition 3. *A meander of order n is decomposable in* MEANDER *if and only if there exists a substring in its formal sequence which begins at index 2 or greater, has odd length between* 3 *and* $2n-3$ *inclusive, and which is an integer interval.*

Proof. A composition of meanders yields a permutation with the substring $w'_2, \ldots, w'_{2k}$, which has the necessary requirements. On the other hand, suppose a meander $[x_1, \ldots, x_n]$ contains such a substring from x_k to x_ℓ. then we claim $[x_1, \ldots, x_k]$ □

Proposition 4. MEANDER *is not finitely generated.*

Proof. Proof by construction □

Let P denote this latter set of indecomposable meanders. Every meander is therefore equal to at least one composition of meanders in P. Our strategy is to enumerate M_n by counting how many ways to compose indecomposable meanders of lesser order to get meanders of order n.

There are two main challenges for this strategy. First, while counting we must make sure to "mod out" by equality so that we do not double-count different expressions for the same meander. Second, we must determine how many indecomposable meanders there are of order n. This paper contains a solution to the first challenge, called *leftmost decompositions*, and explores one approach to solving the second.

4. Leftmost decompositions

Here we define a canonical way to write a decomposable meander as a composition of indecomposable meanders.

Definition 3 (Leftmost expressions)**.** *Given a decomposable meander m, a leftmost expression is obtained in the following way. The meander m may be written as $m' \circ_i w$ for a number of triples (m', i, w). Of these triples, select the smallest i and among the w associated with that i, select the one of largest order, and write $m' \circ_i w$. Now, repeat this step to decompose m' and w. The sequence stops when all meanders in the expression are indecomposable.*

Note that there is a single w of largest order because for a fixed index meander composition is injective. This implies there is exactly one leftmost expression for each decomposable meander. An example is shown in [fig].

If we knew how to count P_n, then M_n could be easily enumerated with well-known tree enumeration methods. This problem remains open, but we can via brute-force methods determine a list of indecomposable meanders up to some finite order n, and enumerate the subclass of meanders corresponding to the suboperad generated by this finite set.

We now count the suboperad of Meander generated by the two (indecomposable) meanders of order two, (1 2 3 4) and (1 4 3 2).

5. Meanders and Odd Blobs

For convenience set $m_1 = (1\ 2\ 3\ 4)$ and $m_2 = (1\ 4\ 3\ 2)$.

Theorem 1. *Let S be the subset of meander expressions generated by m_1, m_2 defined as follows, treating each expression as a plane tree with each vertex labeled by m_1 or m_2. For all vertices v, require:*

Case 1: *Case* $v = m_1$

The middle child of v *is either null or* m_1 *and the right child of* v *is null or* m_2

Case 2: $v = m_2$

The middle child of v *is either null or* m_2 *and the right child of* v *is null or* m_1.

Then each S *contains exactly one leftmost expression of each meander generated by* $\{m_1, m_2\}$.

Proof. WIP ☐

Theorem 2.

$$M_n^s = \begin{cases} 1 & \text{if } n = 1 \\ 2H(n-1) & \text{if } n \geq 2, \end{cases}$$

where $H(n) = 1$ *for* $n = 0, 1$ *and for* $n \geq 2, H(n)$ *is given by the recurrence*

$$\begin{aligned} H(n) = {} & 2\sum_{i=0}^{n-2}\sum_{j=1}^{n-i-2} H(i+1)H(j)H(N-2-i-j) \\ & + \sum_{i=1}^{n-1} H(i)H(n-1-i). \end{aligned}$$

Proof. ☐

6. Prime Meanders and Simple Permutations

The astute reader will notice the similarity between the requirements of permutations corresponding to indecomposable meanders and the definition of *simple permutations*

Definition 4 (Simple Permutations)**.** *WIP*

Acknowledgments

References

[1] M. L. Croix, Approaches to the Enumerative Theory of Meanders.

[2] P. Di Francesco, O. Golinelli, E. Guitter, Meanders and the Temperley-Lieb algebraarXiv:9602025v1.