Surface Separation

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Abstract

How many ways can an orientable surface be cut into k pieces such that every point on the cut is necessary? More formally, how many graphs have an embedding on the genus g torus such that 1) the surface is partitioned into k components, and 2) any subset of the embedding does *not* partition the g torus into k components? We show that answering this question is equivalent to the task of counting graphs that admit rotation systems satisfying two inequalities: one depending on the number of faces in the rotation system's cellular embedding, and one depending on the chromatic number of the cellular embedding's dual graph.

Introduction

Suppose we desire to cut a surface in such a way as to separate it into a fixed number of pieces with no extra or unnecessary cutting. More precisely, for a given surface S, we are interested in determining which multigraphs G admit embeddings $T : G \hookrightarrow S$ such that

1. $S \setminus \mathrm{im}(T)$ has k connected components, and
2. For any proper subgraph $G' \subset G$, $S \setminus T(G')$ has fewer than k connected components.

Definition 1. *An embedding of a graph G into a surface S which satisfies the conditions 1 and 2 above will be called a* minimal k-cut of S. *(If only condition 1 is satisfied, we will call it a* k-cut of S*).*

A simple example helps illustrate the conditions for being a minimal k-cut.

Example 2. *The graph $G = B_2$, a bouquet of two circles, admits a minimal 2-cut embedding on the torus, shown in Figure 0.1. Other embeddings of B_2 may be non-separating, non-minimal 2-cuts, or 3-cuts, as shown in Figure 0.2.*

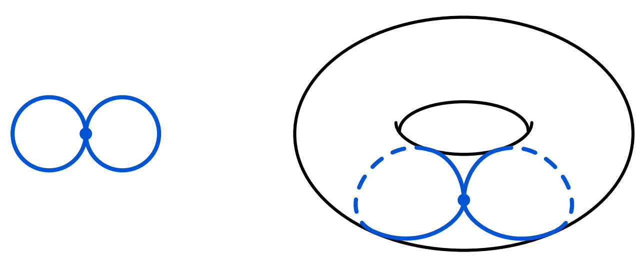

Figure 0.1: The graph B_2 and an embedding giving a minimal 2-cut of the torus.

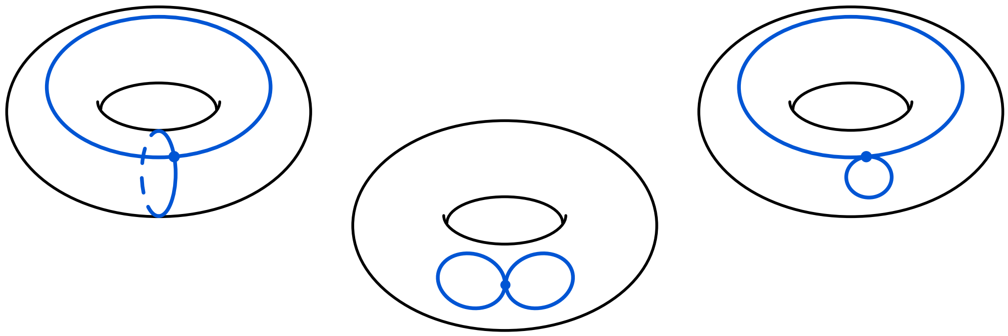

Figure 0.2: Embeddings of B_2 which are respectively non-separating, separate three components, and non-minimal.

In this paper we investigate the construction and enumeration of graphs which admit minimal k-cut embeddings in orientable surfaces. The impetus for considering minimal cuts arose from the work of Bernhard and Veerman, [1, 2], who studied the set of all points on a Riemannian manifold equidistant from two distinguished points, known as a *mediatrix*. These authors proved that a surface mediatrix must be homeomorphic to a finite, closed simplicial 1-complex that produces a minimal 2-cut of the surface. This characterization was then used to show that any mediatrix on the torus of genus one must be homeomorphic to an embedding of one of five graphs shown in Figure **??**. This classification of minimal 2-cuts on the genus 1 torus prompted the consideration of more general k-cuts on genus g tori considered here.

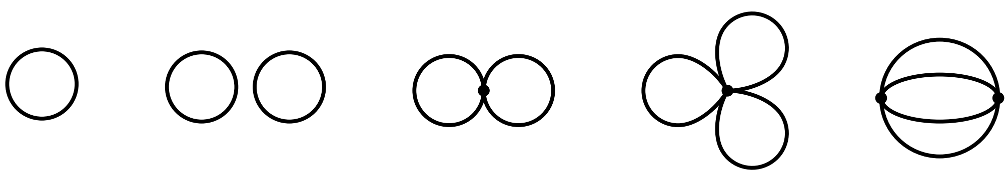

Figure 0.3: Add caption

Previous authors have considered the separating and non-separating properties of graph embeddings, but none appear to have considered minimal k-cuts. Giblin, [4], and Skoviera, [6], investigated graphs which admit non-separating embeddings in a given surface.

Section 1 establishes necessary topological graph theory terminology. Section 2 first provides basic bounds on the size of a graph admitting a minimal k-cut, then an exact combinatorial characterization via rotation systems and their geometric duals.

1 Background

In all that follows, by *graph* we mean a finite multigraph, allowing loops and multiple edges. For a surface S and graph G, an *embedding* of G into S is an injective immersion of (a topological realization) of G into S. The connected components of the complement of the image of an embedding will be called *faces*. An embedding is *cellular* if all of its faces are homeomorphic to open discs and *semi-cellular* if faces are allowed to be open discs with some number of punctures. We will not require our embeddings to be cellular or semi-cellular, so faces of the embedding may contain handles.

1.1 Embeddings

We will pursue enumeration algorithms for a variety of classes of minimal cuts, so it is necessary to discuss combinatorial representations of graph embeddings. Here we extend the canonical representation of a graph embedding, known as a rotation system, to the non-cellular case. We depart slightly from the literature (e.g., [5]) by allowing rotation systems to have multiple connected components.

Definition 3 (Rotation System)**.** *A* rotation system *is a pair of permutations* (α, σ) *on a set* D *such that* α *is an involution and* α *and* $\alpha\sigma$ *are fixed-point-free. The set of orbits of* D *under* $\langle\alpha, \sigma\rangle$ *are the* connected components *of* D*. In the case that* $\langle\alpha, \sigma\rangle$ *acts transitively on* S*, we call the rotation system* connected*.*

Given a rotation system R, we will write G_R to denote its associated graph (where cycles in σ are vertices and edges are drawn between two vertices when they contact a and $\alpha(a)$ in their cycles, respectively). A connected rotation system fully determines a cellular graph embedding.

2 Characterization of minimal k-cuts

Consider a graph G embedded as a minimal k-cut of a closed genus g surface S. Because we are concerned only with topological realizations of G, we assume that all vertices of G have valence greater than or equal to 3, apart from isolated loops (made from a single vertex and edge). Under this assumption, if G contains no isolated loops,

$$e \geq \frac{3}{2}v, \tag{1}$$

where v, e are the number of vertices and edges of G, respectively. If G embeds as a minimal 2-cut and has no isolated loops, we have the stronger bound

$$e \geq 2v, \tag{2}$$

since valence 3 vertices (or odd vertices in general) cannot be part of a minimal 2-cut embedding. The inequalities (1) and (2) will be used in the next section.

2.1 Euler characteristic analysis

Any embedding of G into a surface S determines a ribbon graph R_G, which can be viewed as a subset of S, [3]. Note that R_G and G have the same Euler characteristic, $\chi(R_G) = \chi(G) = v - e$, since G is a deformation retract of R_G. In the case of a minimal k-cut embedding, removing R_G from S and taking the closure of the result produces k surfaces with boundary, $S_1, \dots, S_k$. The equality

$$\chi(S) = \chi(S_1) + \cdots \chi(S_k) + \chi(R_G) \tag{3}$$

then follows from inclusion-exclusion for the Euler characteristic and the fact that the intersections $S_i \cap R_G$ consist of disjoint unions of loops, with Euler characteristic 0. If g_i, n_i denote the genus and number of boundary components, respectively, of surface S_i, we have from (3)

$$e - v = 2(k-1) + 2\left(g - \sum_{i=1}^{k} g_i\right) - \sum_{i=1}^{k} n_i. \tag{4}$$

Combining (4) with inequality (1) yields the bound

$$\frac{1}{2}v \leq 2(k-1) + 2\left(g - \sum_{i=1}^{k} g_i\right) - \sum_{i=1}^{k} n_i \tag{5}$$

for graphs G embedded as k-cuts. In the case of $k = 2$, combining (4) and (2) yields the stronger bound

$$v \leq 2(k-1) + 2\left(g - \sum_{i=1}^{k} g_i\right) - \sum_{i=1}^{k} n_i. \tag{6}$$

We are now prepared to give a crude bound on the size of a graph G that admits a minimal k-cut embedding.

Proposition 4. *If G admits a minimal k-cut of a genus g surface S, then*

$$\frac{1}{2}v \leq k - 2 + 2g, \qquad e \leq v + k - 2 + 2g. \tag{7}$$

When $k = 2$ we have the stronger bound

$$v \leq k - 2 + 2g, \qquad e \leq v + k - 2 + 2g. \tag{8}$$

2.2 Exact requirements for general min-cuts

The familiar concept of a dual graph to a planar graph embedding may be extended to graphs on surfaces in the following way:

Definition 5 (Cycle Dual)**.** *Given a rotation system $R = (\alpha, \beta)$, its* cycle dual *is $R^* = (\beta\alpha, \beta)$.*

Given an initial graph G, the cycle dual for a particular rotation system on G gives rise to a dual graph G'. Our characterization of min-cuts will require analysis of the set of all such dual graphs.

Definition 6 (Dual Set)**.** *Given a graph G, let $\mathbf{R}$ be the set of all rotation systems R such that $G_R = G$. Then the* dual set *of G is $\mathbf{G}^* = \{G_{R^*} : R \in \mathbf{R}\}$*

In what follows we reduce the question whether G admits a min-(g,k)-cut to a question of what sort of graphs exist in $\mathbf{G}^*$. We just need two more tools:

Definition 7 (Rotation system associated with an embedding)**.** *Given an embedding $G \hookrightarrow T$, the* associated rotation system $R_{G \hookrightarrow T}$ *is obtained as follows: thicken G, remove the rest of T, and close the boundaries of the remaining surface with discs. This defines a cellular embedding of G (on some number of surfaces) and by extension the rotation system R.*

Definition 8 (Gluing Graph)**.** *The* gluing plan *of a graph embedding $G \hookrightarrow T$ is a bipartite multigraph where one vertex set is the connected components of G, the other vertex set is the set of regions of $G \hookrightarrow T$, and an edge is placed between connected component v_G and region v_T when they share a boundary.*

Theorem 9. *G admits a (g, k)-cut iff there exists a $G^* \in \mathbf{G}^*$ such that*

$$d \leq v^* - k + 1 \leq 2g - e + v,$$

where v^* is the number of vertices in G^* and d is the number of connected components in G.

Proof. Suppose $G \mapsto T$ is a (g, k)-cut embedding. Let $T_1, \ldots, T_k$ be the regions of T as partitioned by the embedding of G and let R be the rotation system associated with $G \mapsto T$. The genera in T can be grouped as follows: holes in the thickened G, holes cut by the boundaries between the thickened G and the T_is, and holes in the T_is themselves. Let these genera be counted respectively by g_G, g_{cut}, and g_T.

To count g_G, note that the genus of R is equal to g_G. For any rotation system we have

$$\begin{aligned} 2d - 2g_R - f &= v - e \\ \implies g_R &= d - \frac{1}{2}(e - v - f), \end{aligned}$$

where d is the number of connected components in G_R. Letting v^* be the number of vertices in G_{R*},we thus have

$$g_G = d + \frac{1}{2}(v + v^* - e). \tag{9}$$

because by definition there is exactly one vertex in G_{R^*} per face in R.

To count g_{cut}, consider the gluing graph P of $G \mapsto T$. By definition P has $d + k$ vertices and, because there is one boundary for each vertex in G_{R^*}, we have

$$\begin{aligned} 1 - g_{\text{cut}} &= v_P - e_P \\ \implies g_{\text{cut}} &= v^* - (k + d) + 1. \end{aligned} \tag{10}$$

Certainly $g_{\text{cut}} \geq 0$, so (10) gives us the first desired inequality. Combining (9) and (10), we see that the total genus of T is

$$\begin{aligned} g &= \frac{1}{2}(v^* + e - v) - k + 1 + g_T \\ &\geq \frac{1}{2}(v^* + e - v) - k + 1. \end{aligned}$$

and elementary agebraic manipulation completes the implication.

($\Longleftarrow$) Suppose we have a graph G such that $\exists G^* \in \mathbf{G}^*$ such that

$$d \leq v^* - k + 1 \leq 2g - e + v$$
$$\implies g \geq \frac{1}{2}(e - v + v^*) - k + 1 \quad \text{and} \quad d \leq v^* - k + 1.$$

We construct a (g,k)-cut embedding $G \mapsto T$ as follows:

Because $G^* \in \mathbf{G}^*$, there exists an R such that $G^* = G_{R^*}$, so let $G \mapsto S$ be the (perhaps multi-component) ribbon graph embedding defined by R. Further, let T_1 be a surface of genus $g - \frac{1}{2}(v^* - v + e) - k + 1$ with $v^* - k + 1$ boundaries, and let $T_2, \ldots, T_k$ be discs. Select $v^* - k + 1$ boundaries from S such that at least one boundary from each connected component is selected (note that we can do this because $d \leq v^* - k + 1$ by assumption) and glue them to T_1 in a one-to-one fashion (we can do this because the boundary sets are equinumerous). Finally, glue the remaining $V^* - (v^* - k + 1) = k - 1$ boundaries in S to the $k - 1$ discs $T_2, \ldots, T_k$ in a one-to-one fashion.

This construction defines a new embedding $G \mapsto T$. T has no boundary components because the above construction defines an injective function from the equinumerous sets of boundaries in S and boundaries in $T_1 \cup T_2 \cup \cdots \cup T_k$. $G \mapsto T$ is also a k-cut because each region of the ribbon graph is attached to one of the k T_is.

If T is a g-torus then we are done. To determine the number of genera in T, apply the equalities derived during the other direction of this proof:

$$\begin{aligned} g_T &= g_G + g_{\text{cut}} + g_{T_1} + (g_{T_2} + \cdots + g_{T_k}) \\ &= \big(d + \frac{1}{2}(v + v^* - e)\big) + \big(v^* - k - d + 1\big) + \big(g - \frac{1}{2}(v^* - v + e) - k + 1\big) \\ &= g. \end{aligned}$$

□

Theorem 10. *G admits a min-(g,k)-cut iff there exists a $G^* \in \mathbf{G}^*$ such that the condition in the previous theorem holds and*

$$\gamma(G^*) \leq k,$$

where $\gamma(\cdot)$ denotes chromatic number.

Proof. ($\Longrightarrow$) Suppose G admits a min-(g,k)-cut $G \mapsto T$ and let $G^* \in \mathbf{G}^*$ be the graph of the cycle dual of the embedding's associated rotation system. Now produce a coloring of G^* in the following way: assign a different color to each of the k regions, thicken G, then propagate the color assignments of the regions to the boundaries of the thickened G. Finally, for each vertex G^*, assign it the color currently assigned to its associated cycle in the thickened graph.

Is this assignment a proper coloring? By way of contradiction suppose $v_1, v_2 \in G^*$ are adjacent vertices with the same color assignment. Because v_1, v_2 are adjacent, they are associated with cycles in R that share an edge $e \in G$. This means that in $f = G \mapsto T$, $f(e)$ is bordered by the same region on both sides. Thus $f(e)$ is not needed for f to be a (g,k)-cut, contradicting the given that $G \mapsto T$ is a minimal (g,k)-cut.

Observe that we have produced a proper k-coloring for $G^* \in \mathbf{G}^*$, so $\gamma(G^*) \leq k$.

($\Longleftarrow$) Suppose there is a $G^* \in \mathbf{G}^*$ that satisfies the inequalities stipulated by the theorem.

We will now construct an embedding from a minimum vertex coloring. Let one of the vertex colors (the choice is arbitrary) be red and for each connected component of G^* that does not contain a red vertex, replace one color in this component with red. Note that there are now at least d red vertices in G^*. Now select $k - \gamma(G^*)$ vertices such that at least one red vertex in each connected component is unselected and at least one vertex of each other color is also unselected. Thus we are selecting $k - \gamma(G^*)$ vertices from a set of size $v^* - \big(\gamma(G^*) + d - 1\big)$. This is possible because of the given

$$\begin{aligned} d &\leq v^* - k + 1 \\ \implies k - \gamma(G^*) &\leq v^* - \big(\gamma(G^*) + d - 1\big) \end{aligned}$$

Finally, assign each of these selected vertices a new color, distinct from all the rest. Observe that we now have a proper vertex coloring with exactly k colors where every connected component contains a red vertex.

Let $c_1, \ldots, c_k$ be the number of vertices labeled with each of the k colors. Now define T_1 to be a surface of genus $g - \frac{1}{2}(v^* - v + e) - k + 1$ with c_1 boundary components and $T_2, \ldots, T_k$ be spheres with $c_2, \ldots, c_{\gamma(G^*)}$ respective punctures. Color the boundaries of the ribbon graph of R with their associated vertex colors in G^* and for each $i = 1, 2, \ldots, k$ glue boundaries of color i in the ribbon graph to boundaries in T_i in a one-to-one fashion. Observe that by construction the resulting surface T has no boundaries.

It remains to show that T has genus g and that $G \mapsto T$ is a minimal k-cut (it is certainly a k-cut). That T's genus is g follows in precisely the same way as in the previous theorem. To see minimality, suppose to the contrary that there is an unecessary edge $e \in G$ under the mapping $f = G \mapsto T$. This implies $f(e)$ is bordered by the same region on both sides. But by construction and the definition of G^*, this means that the two vertices of G^* that are assoicated with the boundaries on either side of $f(e)$ are the same color. This contradicts the given that we began with a proper coloring. $\square$

These proofs give rise to new "spectrum"-related questions concerning graph embeddings. Certainly $\{v(G^*) : G^* \in \mathbf{G}^*\}$ is an uninterrupted integer interval (because of graph genus spectrum proofy things), but is $\{\gamma(G^*) : G^* \in \mathbf{G}^*\}$ also? Does $\big\{\big(v(G^*), \gamma(G^*)\big) : G^* \in \mathbf{G}^*\big\}$ form a rectangle in $\mathbb{Z}^2$ or some more complicated subset?

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